Q1: For $H_0: \beta_1 = 0$ vs. $H_1: \beta_1 > 0$

test statistic is $t = \dfrac{\hat{\beta}_1}{se(\hat{\beta}_1)}$

why do we reject H_0 when t is very small?

Intuitively, this makes sense. But how to get this from strict mathematical proof?

A1: (Credit goes to Ganghua.)

The family of normal densities has monotone likelihood ratio property. By Thm 12.9 on Keener's book ("Theoretical Topics for a Core Course"),

$$\varphi^*(X) = \begin{cases} 1, & T(X) > c \\ 0, & T(X) < c \end{cases}$$ will be the UMP test,

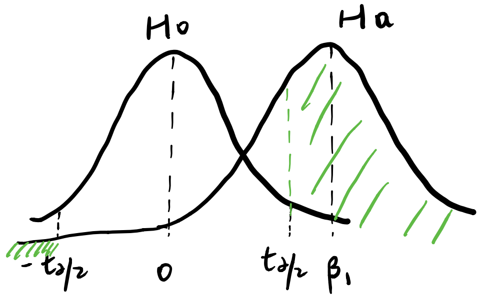
which means uniformly most powerful.

And by checking the density of normal dist, we can show that

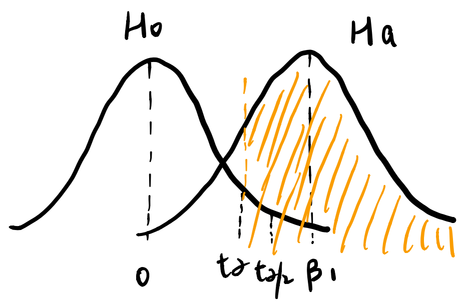
$$\varphi(x) = \begin{cases} 1, & t = \dfrac{\hat{\beta}_1}{se(\hat{\beta})} > c \\ 0, & t < c \end{cases}$$

is equivalent to $\varphi^*(X)$.

Or we can use graphic understanding.



If the rejection region is $\{|t| > t_{\alpha/2}\}$ the power would be the shadowed area ▨.



If the rejection region is $\{t > t_\alpha\}$, the power would then be the shadowed area ▨.

The power in the second case is larger than that in the first case, which then validates the choice of rejection region.