UMVU

Step 1 : $\begin{cases} \text{exponential family} : & \text{Find out } T \text{ as complete ss} \\ \text{involves } 1_{\{T<\theta\}} : & T \text{ as sufficient} \end{cases}$

Step 2 : Check the probability distribution of T

Step 3 : Try to find an unbiased estimator $\delta(X)$.

If discrete, then use conditional probability

If continuous, then use conditional density.

$\mathbb{E}[\delta(X) \mid T]$

Step 4 : If cannot find an unbiased estimator,

If involves $1_{\{T<\theta\}}$ or $1_{\{T>\theta\}}$, then

$\mathbb{E}\delta(T) = g(\theta)$, take derivative.

If not, use Taylor series, $\delta(T) = \sum_{n=0}^{\infty} C_n T^n$,

$\mathbb{E}\delta(T) = g(\theta)$

match the coefs of polynomial terms.

Convergence

① a.s. convergence

(i) Borel - Cantelli lemma

$$A_n = |X_n - X| > \varepsilon$$

$$\sum_n P(A_n) < \infty$$

equivariant estimator

Step 1 : Get the density, find the complete and sufficient statistics T

Step 2 : Try to find $\delta_0(x)$

(i) T

(ii) usual mean, median, MLE

(iii) order statistics

Step 3 : Check the loss

(i) loss is general

$$v^* = \arg\min_{v} \mathbb{E}_0[\rho(\delta_0(X) - v) \mid Y]$$

(ii) loss is squared loss

$$v^* = \mathbb{E}_0[\delta_0(X) \mid Y]$$

(iii) loss is absolute loss

$$v^* = \text{median of } \delta_0(X) \mid Y$$

(iv) if Y is ancillary, $\delta_0(X)$ is T, (Basu's) then $v^* = \arg\min_{v} \mathbb{E}_0[\rho(\delta_0(X) - v)]$

(v) if Y is $n-1$ dimensional, $v^* = c$

Step 4: If $f(x_1-\xi, \cdots, x_n-\xi)$ is easy to get,

(i) general $\rho(\cdot)$

$$\delta^* = \arg\min_{d} \frac{\int \rho(d-u) f(x_1-u, \cdots, x_n-u)\, du}{\int f(x_1-u, \cdots, x_n-u)\, du}$$

(ii) ρ is squared loss

Pitman estimator

$$\delta^*(X) = \frac{\int u f(x_1-u, \cdots, x_n-u)\, du}{\int f(x_1-u, \cdots, x_n-u)\, du}$$

Prove convergence in probability

① Slutsky's theorem

② Markov's inequality. $P(|X| > \varepsilon) \le \frac{E|X|}{\varepsilon}$

③ Chebychev's inequality. $P(|X-\mu| > \varepsilon) \le \frac{Var\,X}{\varepsilon^2}$

EM algorithm

X is the incomplete data, Y is the complete data.

E-step: $Y^{(P)} = E[Y \mid X, \theta^{(P-1)}]$

M-step: under Y, get MLE $\theta = \delta(Y)$

$$\theta^{(P)} = \delta(Y^{(P)})$$

For mixture model, such as the following.

$$X_i \overset{i.i.d}{\sim} \sum_{j=1}^{m} p_j \frac{1}{\sqrt{2\pi\sigma_j^2}} e^{-\frac{(X_i-\mu_j)^2}{2\sigma_j^2}}$$

Include Y, $P(Y=j) = p_j$

then $L(X, y; \theta) = \prod_{i=1}^{n} p_{y_i} \frac{1}{\sqrt{2\pi\sigma_{y_i}^2}} e^{-\frac{(X_i-\mu_{y_i})^2}{2\sigma_{y_i}^2}}$